`alt.algebra.help`
Guidelines and FAQ

Last updated March 24, 2008

This document is available in HTML, PDF, and DVI formats:

http://aah.ryan-usa.com/

http://aah.ryan-usa.com/aah.pdf

http://aah.ryan-usa.com/aah.dvi

Contents

0.1 Who should read this document?

If you plan on participating in `alt.algebra.help` you are encouraged to read this document. Reading this document just might answer your question before you even ask it, and will also inform you of a few other useful things such as how to represent mathematical expressions, what types of topics can be discussed, etc. Not reading it will not cause the end of the world, but it may result in someone having to repeat something that has already been discussed ad infinitum, asking for clarification of your notation, or referencing you back to this document.

Lurkers may benefit from the document as well.

0.2 Introduction

This is the frequently asked questions document for the newsgroup `alt.algebra.help`, last updated March 24, 2008. As the title suggests, the primary purpose of this document is to provide answers to frequently asked questions. Also included are some basic guidelines of participation. These guidelines are not intended to be formal rules that in any way govern the newsgroup. They serve only to identify what have come to be accepted as norms. In this spirit, it will be nice if you attempt to follow these guidelines. Doing so may result in a more enjoyable visit. Not doing so may result in a less enjoyable visit. That's all.

If you are a newcomer to Usenet, see the newsgroup news.announce.newusers for general information. You can get newsgroup access at Giganews Newsgroups (http://www.giganews.com). Also worth looking at is the newsgroup news.newusers.questions, which also has a website at http://www.anta.net/misc/nnq/. These resources, and many others, contain an abundance of information for newcomers to the Usenet community.

Contributions or suggestions for this document are welcome at aah@ryan-usa.com. Please do not use this address to ask for help with algebra; that's what the newsgroup is for. Such email will be returned to sender, or simply ignored.

0.3 Available file formats of this FAQ

This document is currently available in the following formats:

- HTML, located at http://aah.ryan-usa.com/
- PDF, located at http://aah.ryan-usa.com/aah.pdf
- DVI, located at http://aah.ryan-usa.com/aah.dvi

The HTML version of this document will produce less than ideal quality hard copy, especially of pages containing graphical expressions (e.g. the sections on typing mathematical expressions, frequently asked questions, and reference.) The HTML version has been optimized for fast loading into a browser, not for good quality printouts. For printing, you will be much better served to download the .pdf or .dvi.

0.4 Revision history

- **March 24, 2008**
  - Updated the Introduction.
  - Added restrictions to the laws of exponents and radicals.
  - Updated the Inverse Trigonometric Functions page.
- **July 17, 2006**
  - Updated the links page.
- **November 14, 2005**
  - Corrected a typographical error. No new or updated material.
- **May 31, 2005**
  - Added a Reference Center containing common facts and formulas.
  - Added "Is defined to be equal to" to the Basic Operations, Relations page.
- **May 20, 2003**
  - URL changed to http://aah.ryan-usa.com/.
  - Added section "Can I post the same question to other newsgroups in addition to `alt.algebra.help`?"
  - Removed section "Needed contributions."
- **October 21, 2002**
  - Added section "Where does the Quadratic Formula come from?"
  - Minor format changes to the PDF.
- **October 7, 2002**
  - Added section on needed contributions
  - Added section on importance of parentheses
  - Added section on order of operations
  - Added section "Is -1^2 equal to 1 or -1?"
  - Minor updates to tables on how to type expressions
  - Minor verbiage modifications throughout
  - General Guidelines section now included in periodic posting to alt.algebra.help, alt.answers, news.answers
- **January 10, 2002**
  - Initial document.

0.5 Copyright notice and credits

0.5.1 Credits

Thanks to these people for their suggestions:

- David W. Cantrell
- Doug Magnoli
- Mark Thakkar
- Bob Bruner
- Brian M. Scott
- William Elliot
- Ron Verrall
- Vincent Johns
- R. Dan Henry

...and a special thanks to Joe Chacko, who wrote the original FAQ upon which this document was inspired.

Chapter 1

General Guidelines

1.1 Who should use alt.algebra.help?

Anyone seeking (or providing) help with any topic related to algebra. This may include students, parents, tutors, teachers, or anyone else having an interest in algebra. This newsgroup is not moderated, so various others who have no business here (e.g. spammers,) will also post articles asking for help of a different kind. Ignore them, killfile them, report them to a proper authority, do whatever you wish with them–but please do not respond to them in the newsgroup. Doing so only further decreases the signal-to-noise ratio.

1.2 What can be discussed on alt.algebra.help?

As the name implies, `alt.algebra.help` exists primarily for the purpose of seeking help with algebra. Anything remotely related to algebra in some form or fashion is usually welcome here. Specific topics suitable for discussion include, but are not limited to, anything covered in a jr. high school, middle school, high school, or undergraduate algebra course, or their equivalent in countries other than the U.S. This does not necessarily mean that only "algebra" in the high school sense of the word is discussed. Other areas of mathematics are also discussed with varying frequency, e.g. geometry, trigonometry, calculus, probability and statistics, linear algebra, abstract algebra, and number theory, just to name a few. Educators also share teaching practices and experiences. Discussions concerning calculators and their (ab)use also occur with some regularity.

These are just some of the topics that are suitable for discussion here. While some areas of these subjects (or other higher mathematics) may be better suited for other newsgroups, as a general rule if there is someone listening here who can help you with your question, chances are they will. And, usually there is someone listening.

1.3 What general format should my article adhere to?

- For best overall results and readability, try to use a simple fixed-width font (e.g. Courier New). Also ensure your newsreader is configured to post in plain text, not HTML.
- Don't ask questions that are too broad, e.g. "Can someone help me with algebra?" Try to narrow your question to the particular topic or process you are concerned with, e.g. "Can someone explain to me the process for solving a quadratic equation by factoring?"
- Use a descriptive subject line. Readers will have a general idea of what your message pertains to prior to downloading (and having to read) the message body.

  Don't do:

  ```
  Subject: HELP!
  ```

  Do:

  ```
  Subject: Solving quadratic equations by factoring
  ```

- Clearly explain the problem and include specific instructions, whether they be to solve, simplify, etc. If possible, try to reproduce the instructions to the problem exactly as they were given to you. Also, consider telling the group the level of math you are at. There may be (err. . . will be) different methods of approaching your problem, so if the group knows your particular competency level (grade, course, etc.,) they can formulate a response suitable for that level.

- Place math expressions on a single line if possible. Your expression may not "line up" the same way on all newsreaders, especially after being quoted multiple times. These problems can be minimized by placing your expressions on a single line.

  Don't do:

  ```
   x+3
  -----
    2
  ```

  Do:

  ```
  (x+3)/2
  ```

- If you need to use multiple lines (e.g. a passage showing the individual steps of solving an equation, listing a matrix, etc.) be sure to always begin each line of the object or passage at the beginning of the "line" in your editor (no preceding spaces,) and end each line with a hard return. In general, anywhere there is a math expression on a line by itself, that line should end with a hard return. Use "in-line" expressions sparingly (math expressions that are side-by-side or interspersed with normal text,) and only for very short expressions that can survive line-wrapping and quoting with minimal distraction to the reader.

  Don't do:

  ```
  Here's how I proceeded in solving this equation:  x^2 + 5x
  + 7 = 1, x^2 + 5x + 6 = 0 ...got 0 on one side, (x+2)
  (x+3) = 0 ...factored, x = 2, 3.  The answer key says the
  correct answers are x = -2, -3.  Where did I go wrong?
  ```

  Do:

  ```
  Here's how I proceeded in solving this equation:

  x^2 + 5x + 7 = 1

  x^2 + 5x + 6 = 0    ...got 0 on one side

  (x+2)(x+3) = 0      ...factored

  x = 2, 3

  The answer key says the correct answers are x = -2, -3.
  Where did I go wrong?
  ```

- Be sure to explain the specific step(s) you are having trouble with and include your attempt(s), even if you know they are wrong. You will receive more useful help if you do this.

- Use plenty of parentheses, brackets, etc. if an expression may otherwise be interpreted in more than one way.

  Don't do:

```
x+3/2
```

Do:

```
(x+3)/2     or     x+(3/2)
```

Don't Do:

```
x^3c+7
```

Do:

```
x^(3c) + 7     or     x^(3c+7)
```

For more information, see the section on importance of parentheses.

1.4 Can I post the same question to other newsgroups in addition to `alt.algebra.help`?

Of course, assuming the subject matter is on-topic for the other newsgroups (consult the FAQ or charter for the other newsgroups to see.) There is, however, a right way and a wrong way to go about it. The wrong way is to send separate posts to each newsgroup. The right way is to "cross-post," meaning to include all the newsgroups on the Newsgroups: line of a **single** post. By cross-posting, replies in one newsgroup will automatically be sent by default to all the other newsgroups the original post was addressed to. If you don't cross-post but instead send separate posts to each newsgroup, a reply in one newsgroup is posted *just to that newsgroup* and not the others. This is considered poor netiquette, so please don't do it. Although it may not bother the original poster, it can be a big inconvenience for those replying to the post. It can be very frustrating taking the time and effort to post a detailed response, only to learn later the question has already been answered basically the same way in another newsgroup (perhaps the person does not subscribe to all the newsgroups). This situation can be avoided by properly cross-posting your question. It's better for the original poster too, since he need follow only one of the newsgroups to see the responses from all of the newsgroups.

Also, you should cross-post to only a very few newsgroups (say, two or three.) Addressing too many newsgroups may result in the post being filtered out by way of a personal killfile or similar mechanism (for instance, a spam filter at the server level).

1.5 Can I get (or provide) help with a homework assignment?

If you are a student you can get excellent assistance here, but don't expect too much if you just want someone to do your homework for you. If you post something like "I need answers to these problems...NOW" or the like, you will rarely get what you ask for. Several regular contributors are either professional educators, or at the very least concerned others who enjoy helping others while keeping the muscle between the ears limber. Most would rather help you understand a process, as opposed to just cranking out an answer to your problem.

Don't get the wrong impression from the above paragraph. If you are having trouble with a problem on an assignment, or any other algebra problem for that matter, don't hesitate to ask for help. This is what the group is for. However, you are more likely to receive useful responses if you explain specifically what you do not understand about the problem, include your attempt at solving it, and ask for specific guidance with the process for arriving at the answer. This indicates to potential responders that you have a sincere interest in knowing *how* to do the problem, as opposed to giving the possible impression that all you want is the answer. Additionally, letting others know specifically where you are having trouble will probably lead to a more useful response that is tailored to your needs.

If you follow these guidelines, who knows, you might just get the answer as well. Ultimately, it is a decision made by the people responding to your inquiry as to how much detail they provide. Some prefer providing just enough detail to steer you in the right direction, for very good reason. Others may offer more detail, which may or may not include the answer to the problem. Even if someone does give you the answer, it is the *process* you are expected to focus on. Giving an answer without also giving some explanation of the process certainly does not "help," and is rarely seen here. When it does occur it is usually frowned upon, so please think twice before giving an answer without any explanation how it was arrived at.

1.6 Can I include binary files?

This is a "plain text only" newsgroup. Please do not attach binaries. If you want to show the group a graphic, HTML document, or other type of richly formatted content, consider placing it on your web server and providing the URL within your article.

Don't do:

```
Consider the attached bitmap of the graph of f(x).
```

Do:

```
Consider the graph of f(x) at http://www.yourisp.net/~username/graph.bmp
```

Chapter 2

Typing Mathematical Expressions on `alt.algebra.help`

The following tables describe how to type commonly used mathematical expressions in plain text, along with usage examples (including the typeset equivalent.) If the expression you need is not listed, please define your own and explain the meaning of your notation. Some of the content of these tables is taken or derived from Joe Chacko's FAQ of 31 July 1996, with permission.

2.1 Commonly used letters

This table describes how to type commonly used letters, along with usage examples (including the typeset equivalent.) If the expression you need is not listed, please define your own and explain the meaning of your notation.

Expression	Description	Examples
`x`	A variable, or unknown. The letter x is used most commonly for an unknown, although there are a few conventions, as follows, for other letters. Try to use only lowercase letters for variables wherever possible. Uppercase letters are many times used for matrices, sets, and other things.	Let x be the number of tickets sold for the event... `Let x be the number of tickets sold for the event...`
`x,y`	Real numbers.	Let x and y be any two real numbers... `Let x and y be any two real numbers...`
`u,v`	Vectors.	For vectors $\mathbf{u} = \langle a, b \rangle$ and $\mathbf{v} = \langle c, d \rangle \ldots$ `For vectors u=<a,b> and v=<c,d>...`
`n`	Natural number.	If true for n, prove true for $n+1 \ldots$ `If true for n, prove true for n+1...`
`q,r`	Rational numbers.	For all rational numbers q and $r \ldots$ `For all rational numbers q and r...`
`i`	The imaginary unit, i.	$i^2 = -1$ `i^2 = -1`
`z`	Complex number.	Let z be a complex number of the form $x + iy \ldots$ `Let z be a complex number of the form x+iy...`

Expression	Description	Examples
`f,g,h`	Functions.	Let f be a function defined by $f(x) = x^3$. `Let f be a function defined by f(x)=x^3.`
`e`	The constant e, the base of the natural logarithm.	$\ln x = \log_e x$ `ln(x) = log_e (x)`
Greek letters	In most cases, the intended “case” of a Greek letter can be inferred from the context.	α, β, γ, δ, ϵ, θ, etc. `alpha, beta, gamma, delta, epsilon, theta, etc.`
	To explicitly indicate the uppercase, type the first letter in uppercase.	Γ, Δ, Θ, Λ, Ξ, Π, Σ, etc. `Gamma, Delta, Theta, Lambda, Xi, Pi, Sigma, etc.`

2.2 Basic operations, relations

This table describes how to type basic operation and relation symbols, along with usage examples (including the typeset equivalent.) If the expression you need is not listed, please define your own and explain the meaning of your notation.

Expression	Description	Examples
`+`	Addition	$2 + 3 = 5$ `2 + 3 = 5`
`-`	Subtraction / Negation	$2 - 3 = -1$ `2 - 3 = -1`
`*`	Multiplication	$2 \times 3 = 6$ $2 \cdot 3 = 6$ `2 * 3 = 6`

	Note—Multiplication is also implied by juxtaposition.	$xy = x \cdot y$
		$3x = 3 \cdot x$
		
		`xy = x*y`
		`3x = 3*x`

Expression	Description	Examples
`/`	Division	$\frac{4}{2} = 2$ $4 \div 2 = 2$ `4/2 = 2`
`=`	Equal	$x = 1$ `x = 1`
`:=`	Is defined to be equal to	$\pi \equiv \frac{C}{D}$ $\pi \triangleq \frac{C}{D}$ $\pi := \frac{C}{D}$ `pi := C/D`
`~`	Approximately equal	$\pi \approx \frac{22}{7}$ `pi ~ 22/7`

`<>`
 	
or
 	
`!=`
	Not equal	$4 \neq x + 2$, for $x \neq 2$
		
		`4 <> x+2, for x <> 2`
		`4 != x+2, for x != 2`

`>`	Greater than	$4 > 2$ `4 > 2`
`<`	Less than	$2 < 4$ `2 < 4`
`>=`	Greater than or equal	$2 + x \geq 4$, for $x \geq 2$ `2+x >= 4, for x >= 2`
`=<`	Less than or equal	$4 \leq x + 2$, for $2 \leq x$ `4 =< x+2, for 2 =< x`
`+/-`	Plus or minus. Indicates the positive and negative.	$x = \pm 1 \;\Rightarrow\; x^2 = 1$ `x = (+/-)1 ==> x^2 =1`
`^`	Exponentiation	$2^2 = 4$ `2^2 = 4`

Expression	Description	Examples

`sqrt()`	Denotes the principal (nonnegative) square root of the contents of ().
	
	Use +/- to denote both roots.
		$\sqrt{4} = 2$
		$\pm\sqrt{4} = \pm 2$
		
		`sqrt(4) = 2`
		`(+/-)sqrt(4) = (+/-)2`

`cbrt(m)`	Denotes the cube root of m	$\sqrt[3]{8} = 2$ `cbrt(8) = 2`
`[n]root(m)`	Denotes the nth root of m	$\sqrt[5]{32} = 2$ `[5]root(32) = 2`
`!`	Factorial	$3! = 3 \times 2 \times 1$ `3! = 3*2*1`
`:`	to (ratio)	`The ratio of boys to girls is 4:5`

`|x|`
 	
`|z|, mod z`
 	
`|v|`
	This notation is used for several related concepts.
	
	For a real number `x` it denotes the absolute value of `x`, where `|x|=sqrt(x^2)`.
	
	For a complex number `z=x+iy` it denotes the modulus of `z` (so does `mod z`), where `|z|=sqrt(x^2+y^2)`.
	
	It also denotes the length, or norm, of a vector `v=<x,y>`, where `|v|=sqrt(x^2+y^2)`.
	
	The norm of `v` can also be denoted by `||v||`, as described in a later table.
		$\|-7\| = 7$
		
		$z = 2 - 3i,\ \|z\| = \sqrt{13}$
		
		$\mathbf{v} = \langle -5, 12 \rangle,\ \|\mathbf{v}\| = 13.$
		
		`|-7| = 7`
		
		`z=2-3i, |z|=sqrt(13)`
		
		`v=<-5,12>, |v|=13`

`arg(z)`	For a complex number of the form `z = r[cos(theta)+i*sin(theta)]`, denotes the argument of `z` where `arg(z)=theta`.	$z = 2(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})$
		$\arg z = \frac{\pi}{3}$
		
		`z=2[cos(pi/3)+i*sin(pi/3)]`
		`arg(z) = pi/3`

Expression	Description	Examples

`[[x]]`
 	
or
 	
`floor(x)`
	Denotes the greatest integer of x (also called `floor x`), where `[[x]]` = the largest integer $\leq x$.	$[[-3.2]] = -4$
		$[[3.2]] = 3$
		$\lfloor -3.2 \rfloor = -4$
		$\lfloor 3.2 \rfloor = 3$
		
		`[[-3.2]] = -4`
		`[[3.2]] = 3`
		`floor(-3.2) = -4`
		`floor(3.2) = 3`

`|`	Divides	$2 \mid 4$ `2|4 (2 divides 4)`
`==>`	Implies	$x \geq 4 \Rightarrow x > 2$ `x >= 4 ==> x > 2`
`<==`	Is implied by	$x > 2 \Leftarrow x \geq 4$ `x > 2 <== x >= 4`

`<==>`
 	
or
 	
`iff`
	Implies and is implied by.
	
	If and only if.
		$x = 2 \iff 2x = 4$
		
		`x = 2 <==> 2x = 4`
		`x = 2 iff 2x = 4`

2.3 Logarithms

This table describes how to type logarithmic expressions, along with usage examples (including the typeset equivalent.) If the expression you need is not listed, please define your own and explain the meaning of your notation.

Expression	Description	Examples
`ln(x)`	The natural logarithm, base e, of x.	$\ln x = y \iff e^y = x$ `ln(x)=y <==> e^y=x`
`log(x)`	The common logarithm, base 10, of x.	$\log x = y \iff 10^y = x$ `log(x)=y <==> 10^y=x`
`log_b(x)`	The logarithm, base b, of x.	$\log_{\mathrm{b}} x = y \iff \mathrm{b}^y = x$ `log_b(x)=y <==> b^y=x`

2.4 Trigonometric and hyperbolic functions

This table describes how to type trigonometric and hyperbolic functions, along with usage examples (including the typeset equivalent.) If the expression you need is not listed, please define your own and explain the meaning of your notation.

Expression	Description	Examples

`sin(x) cos(x) tan(x)`
`csc(x) sec(x) cot(x)`
	The trigonometric functions, x in radians unless otherwise stated.	$\sin\frac{\pi}{2} = 1$
		$\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$
		
		`sin(pi/2)=1`
		`cos(pi/4)=[sqrt(2)]/2`

`arcsin(x) arccos(x), etc.`
 	
`or`
 	
`sin^-1(x) cos^-1(x), etc.`
	The inverse trigonometric functions.	$\arcsin 1 = \frac{\pi}{2}$
		$\arccos\frac{\sqrt{2}}{2} = \frac{\pi}{4}$
		
		`arcsin(1)=(pi/2)`
		`arccos[sqrt(2)/2]=(pi/4)`

`sinh(x) cosh(x) tanh(x)`
`csch(x) sech(x) coth(x)`
	The hyperbolic functions.	$\sinh x = \frac{e^x - e^{-x}}{2}$
		$\cosh x = \frac{e^x + e^{-x}}{2}$
		
		`sinh(x)=(e^x - e^-x)/2`
		`cosh(x)=(e^x + e^-x)/2`

`sinh^-1(x) cosh^-1(x), etc.`	The inverse hyperbolic functions.	`sinh^-1(x)=ln[x+sqrt(x^2+1)]`
		
		`cosh^-1(x)=ln[x+sqrt(x^2-1)]`

2.5 Calculus

This table describes how to type calculus expressions, along with usage examples (including the typeset equivalent.) If the expression you need is not listed, please define your own and explain the meaning of your notation.

Expression	Description	Examples

`{a_n} =`
`{a_1,a_2,...,a_n,...}`
	An infinite sequence.	$\{3 + (-1)^n\}$
		$= \{2, 4, 2, 4, \cdots\}$
		
		`{3+(-1)^n}`
		`= {2,4,2,4,...}`

Expression	Description	Examples

`sum[n=1,oo] a_n`
`= a_1+a_2+...+a_n+...`
	An infinite series.	$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$$
		`sum[n=1,oo] 1/(2^n)`
		`= (1/2)+(1/4)+(1/8)+...`

`lim x-->c f(x)`	Limit of f(x) as x approaches c.	$$\lim_{x \to 0} x^2 = 0$$ `lim x-->0 (x^2) = 0`

`dx , dy`	Differentials.
	
	The differential of x.
	The differential of y.
		dx
		dy
		
		`dx`
		`dy`

`y'`	y prime. The first derivative of y.	$y = x^2$
		$y' = 2x$
		
		`y=x^2`
		`y'=2x`

`y''`	The second derivative of y.	$y = x^2$
		$y'' = 2$
		
		`y=x^2`
		`y''=2`

`y'''`	The third derivative of y.	$y = x^2$
		$y''' = 0$
		
		`y=x^2`
		`y'''=0`

`f'(x)`	f prime of x, the first derivative of f.	$f(x) = x^2$
		$f'(x) = 2x$
		
		`f(x)=x^2`
		`f'(x)=2x`

`f''(x)`	The second derivative of f.	$f(x) = x^2$
		$f''(x) = 2$
		
		`f(x)=x^2`
		`f''(x)=2`

Expression	Description	Examples

`f^(n)(x)`	Higher order derivatives. The n^{th} derivative of f.	$f(x) = x^3$
		$f^{(4)}(x) = 0$
		
		`f(x)=x^3`
		`f^(4)(x)=0`

`dy/dx`
 	
`d/dx[f(x)]`
	Another commonly used notation for derivatives. The first is read "the derivative of y with respect to x."	$y = x^2$
		$\frac{dy}{dx} = 2x$
		$\frac{d}{dx}[x^2] = 2x$
		
		`y=x^2`
		`dy/dx = 2x`
		`d/dx[x^2] = 2x`

`(d^2y)/(dx^2)`	The second derivative of y with respect to x.	$y = x^2$
		$\frac{d^2y}{dx^2} = 2$
		
		`y=x^2`
		`(d^2y)/(dx^2) = 2`

`(d^ny)/(dx^n)`	Higher order derivatives. The n^{th} derivative of y with respect to x.	$y = x^3$
		$\frac{d^4y}{dx^4} = 0$
		
		`y=x^3`
		`(d^4y)/(dx^4) = 0`

`pz/px`
 	
`p/px [f(x,y)]`
	Partial derivatives. The first is read "the partial of z with respect to x."	$z = x^2 + xy$
		$\frac{\partial z}{\partial x} = 2x + y$
		$\frac{\partial}{\partial x}\left[x^2 + xy\right] = 2x + y$
		
		`z = x^2 + xy`
		`pz/px = 2x+y`
		`p/px [x^2 + xy] = 2x+y`

`F`	An antiderivative of f.	$f(x) = 3x^2$
		$F(x) = x^3$
		
		`f(x) = 3x^2`
		`F(x) = x^3`

`int(f(x)) dx`	Indefinite integration.	$\int x^2\,dx = \frac{x^3}{3} + C$
		
		`int(x^2) dx`
		`= (x^3)/3 + C`

Expression	Description	Examples

`int[a,b](f(x)) dx`	Definite integration.	$$\int_0^2 x^2\,dx = \frac{x^3}{3}\Big]_0^2$$ $$= \frac{8}{3}$$ `int[0,2](x^2) dx`
		`= (x^3)/3 [0,2]`
		`= 8/3`

`int[a,b]int[c,d]( f(x,y) ) dydx`	Multiple integration. A double integral, in this case.	$$\int_0^1\int_0^2 (x+y)\,dy\,dx$$ $$\int_0^{\frac{\pi}{2}}\int_0^{2\cos\theta} r\,dr\,d\theta$$ `int[0,1]int[0,2] (x+y) dydx` `int[0,(pi/2)]int[0,2 cos(theta)] r drd(theta)`

2.6 Matrices, vectors, and miscellaneous

This table describes how to type matrices, vectors, and other miscellaneous expressions, along with usage examples (including the typeset equivalent.) If the expression you need is not listed, please define your own and explain the meaning of your notation.

Expression	Description	Examples

`[A B C]`
`[D E F]`
`[G H I]`
	A matrix. Enclose each row within [ ].	$$\begin{bmatrix} 2 & 1 & 0 \\ 4 & 7 & 3 \\ 2 & 0 & 1 \end{bmatrix}$$ `[2 1 0]`
		`[4 7 3]`
		`[2 0 1]`

`-->PQ`	A vector, with initial point P and terminal point Q.	$\overrightarrow{PQ}$ `-->PQ`

Expression	Description	Examples
`u = <x,y>` `v = <x,y>`	Vectors **u** and **v** in component form. Component form is defined with initial point at the origin, and terminal point (x,y).	$\mathbf{v} = \langle -5, 12 \rangle$ `v = <-5,12>`
`||v||`	The norm, or length, of vector **v**. If $\mathbf{v} = \langle v_1, v_2 \rangle$, $\\|\mathbf{v}\\| = \sqrt{{v_1}^2 + {v_2}^2}$	If $\mathbf{v} = \langle -5, 12 \rangle$, $\\|\mathbf{v}\\| = 13$ `If v = <-5,12>, ||v|| = 13`

`u dot v`
 	
or
 	
`u .  v`
	Dot product. Only use `u .  v` if there can be no ambiguity with a period or decimal point.	$\mathbf{u} \cdot \mathbf{v}$
		
		`u dot v`
		`u .  v`

`u cross v`
 	
or
 	
`u x v`
	Cross product. Only use `u x v` if there can be no ambiguity with the letter x or the ordinary multiplication operator.	$\mathbf{u} \times \mathbf{v}$
		
		`u cross v`
		`u x v`

`oo, -oo`	Positive infinity, negative infinity	$$\lim_{x \to -\infty} x^2 = \infty$$ `lim x-->-oo x^2 = oo`

2.7 Importance of using parentheses on `alt.algebra.help`

Although incorrect, it is not uncommon on `alt.algebra.help` for someone to transcribe something similar in form to $\frac{x+1}{x-1}$ as:

```
x+1/x-1.
```

Since this is a fairly common issue, many will ask, "Do you mean `x+(1/x)-1` or do you mean `(x+1)/(x-1)`? In other words, it will not necessarily be assumed by everyone that what is written is what actually is *intended*.

Let $x = 2$. Does the above properly evaluate to 3 as intended? Division (denoted by `/`) has precedence over addition and subtraction, therefore this expression is properly evaluated by performing the division $1 \div 2$, adding the result to 2, then subtracting 1 from that result, resulting in $\frac{3}{2}$. In many cases, this is not what is intended when the statement is "written" on `alt.algebra.help`. Even if it *was* the intention, many here will ask for clarification if not otherwise clear from the context.

If $\frac{x+1}{x-1}$ is intended, it is necessary to use parentheses when transcribing this and similar expressions on `alt.algebra.help`. The original fraction can be clearly, and properly, transcribed as:

```
(x+1)/(x-1).
```

Not only does this properly evaluate as intended, it also erases any reasonable doubt one may otherwise have of your intention.

What if your intention really *is* `x+1/x-1`? Although not technically necessary, it is highly recommended that you use parentheses, as in:

```
x+(1/x)-1.
```

The liberal use of parentheses here not only adds clarity but also indicates this really *is* your intention, and should prevent others from asking for clarification.

Similar issues arise while transcribing expressions involving exponents. For example, it would be incorrect to transcribe what appears as x^{y+1} by writing:

```
x^y+1.
```

This is really saying, "Raise `x` to the power of `y`, then add 1," which is considerably different from what x^{y+1} is saying. What we really want to say is, "Raise `x` to the `y+1` power." This can be properly transcribed as:

```
x^(y+1).
```

What if your intention really *is* `x^y+1`? Depending on the context it may add clarity if you use parentheses anyway, as in:

```
(x^y)+1.
```

The liberal use of parentheses here not only adds clarity but also indicates this really *is* your intention, and should prevent others from asking for clarification. Note that in some contexts the intent is clear even without parentheses, as in:

```
ax^2+bx+c=0.
```

Here, it is clear from the context (standard quadratic form) that we definitely do *not* intend:

```
ax^(2+bx+c)=0
```

...but rather `ax^(2) + bx + c = 0.`

Note that it is possible to overdo it. For example, it doesn't add any clarity to write:

```
x^([(y)+1])
```

...as opposed to `x^(y+1).` Rather, the superfluous parentheses and brackets can arguably distract the reader.

In short, use parentheses and brackets if needed to make your intent clear. Otherwise, readers of your question may make certain assumptions that may be inaccurate, or may need to ask for clarification. Either may delay a useful answer to your question.

Chapter 3

Frequently Asked Questions

The primary purpose of this FAQ is not to teach anyone mathematics—that's what the newsgroup is for. It would be unfortunate if the majority of replies in this newsgroup were reduced to a mere reference to a FAQ. However, there are certain questions that have been discussed ad infinitum that deserve some mention here. After all, that's what a FAQ is for—frequently asked questions.

3.1 Does anyone know of a good math website?

Of all frequently asked questions, this one is probably the most frequently asked. There are plenty of sites around that have tutorials and other information on algebra and other math. Here are a just a few to get you started. Most have links to many other sites.

- MathWorld (http://mathworld.com.) A comprehensive on-line mathematics encyclopedia by Eric Weisstein.
- S.O.S. MATHematics (http://sosmath.com), by MathMedics. Review material ranging from algebra to differential equations.
- Purplemath (http://purplemath.com), by Elizabeth Stapel. Practical algebra lessons, links, and more.
- The Mathematical Atlas (http://www.math-atlas.org/welcome.html): a gateway to the fields of modern mathematics.
- Frequently Asked Questions in Mathematics (http://www.cs.uwaterloo.ca/~alopez-o/math-faq/), a.k.a the sci.math FAQ.
- Euclid's *Elements* (http://aleph0.clarku.edu/~djoyce/java/elements/elements.html.) David E. Joyce makes the *Elements* come alive with Java.
- The MacTutor History of Mathematics (http://www-groups.dcs.st-and.ac.uk/~history.)
- Interactive Real Analysis (http://web01.shu.edu/projects/reals.)
- The Math Forum @ Drexel (http://mathforum.org.) Home of Ask Dr. Math, Teacher2Teacher, and several other goodies. Contains links to many sites providing lessons in mathematics, and links for mathematics software.
- The Mathematics Archives (http://archives.math.utk.edu), at the University of Tennessee. A collection of math resources on the web, not only for students but for educators and mathematicians as well. Also contains a collection of math software.
- Visual Calculus (http://archives.math.utk.edu/visual.calculus.) Incorporates technology (interactive Javascript, LiveMath, etc.) as a supplement to text based tutorials. Topics range from pre-calculus to 2nd semester calculus.
- Karl's Calculus Tutor (http://karlscalculus.org), by Karl Hahn. Detailed, easy to follow explanations of many calculus topics. Discussion forum.

3.2 Does anyone know where I can find math software?

The Mathematics Archives (http://archives.math.utk.edu/software.html) at the University of Tennessee. Many freeware/shareware titles, plus demo versions of commercial packages. Contains a large list of links to other math related software sites.

3.3 Why does a negative times a negative equal a positive?

Short answer: We make it this way because it makes life easier. For example, consider the distributive property that we have all grown to love:

$$a(b + c) = ab + ac$$

If a negative times a negative were to equal a negative, this property would not hold for negative numbers, which is a bad thing. Let $a = -1, b = 1, c = -1$ and $(-1)(-1) = -1$:

$$\begin{aligned} -1(1 + -1) &= (-1)(1) + (-1)(-1) \\ -1(0) &= -1 + -1 \\ 0 &\neq -2 \end{aligned}$$

For a longer answer, see this Ask Dr. Math FAQ entry (http://mathforum.org/dr.math/faq/faq.negxneg.html).

3.4 Is -1^2 equal to 1 or -1?

Short answer: It's -1, since exponentiation takes precedence over the implied multiplication (see the section on order of operations.)

Long answer: For some, that simple explanation is lacking because they argue it is the quantity -1 that is being squared, as opposed to 1 that is being squared *then* multiplying the result by -1. A good question to ask those making this claim is, "Why do you associate the negative sign with just the 1, as opposed to associating the negative sign with 1^2?" Then, it usually becomes clear. Claiming the negative sign is somehow "attached" to just the 1, *then* squaring this quantity (yielding 1,) is the same as claiming the implied multiplication has a higher precedence than the exponentiation. The order of operations agreement makes clear that in fact the opposite is true—exponentiation has a higher precedence than multiplication.

Another way of interpreting -1^2 is $0 - 1^2$. The negative sign denotes subtraction here. This is a perfectly legitimate interpretation, and the order of operations agreement tells us this is still -1 because exponentiation takes precedence over subtraction. However, it is worth noting that subtraction is usually defined in terms of addition of the negative quantity. Looking at it this way, this is really the same argument given in the previous and following interpretations.

Yet another way of interpreting the negative sign in -1^2 is that it denotes the unary operation of negation. In the most commonly used order of operations agreement, unary operations are not specifically mentioned since they are usually thought of as being implied by the binary operations, e.g. the unary operation of negation is thought of as being equivalent to the binary operation of multiplying by -1. It is worth mentioning that *some* alternative order of operations agreements actually do give unary operators higher precedence than binary operators. One example of such an agreement is the one implemented in Microsoft Excel (http://support.microsoft.com/support/kb/articles/q132/6/86.asp).

So, technically, it depends on the context (i.e. what convention is in place) as to whether -1^2 is 1 or -1. As explained above, using the most common order of operations agreement, -1^2 is perfectly unambiguous. It's -1, period. However, it is important to realize that any potential confusion should be avoided wherever it can be foreseen. On this forum, as a practical matter, it wouldn't hurt to express -1^2 as something like `-(1^2)` wherever it is feasible to do so for the simple reason that someone *may* find the former to be unclear. Perhaps they use Excel extensively :-) or they do not always assume what you literally type is what you literally mean (which, BTW, is often the case on `alt.algebra.help`).

In short, $-1^2 = -1$ in *most* contexts, especially in the context of "algebra," since most use an order of operations consistent with the one listed in this FAQ, unless advised explicitly on a case-by-case basis that a different agreement is in use.

3.5 What's wrong with this proof of 1 = 2?

Obviously 1 does not equal 2, but this "proof" is discussed from time to time:

$$\begin{aligned} a &= b & & \\ ab &= b^2 & & \text{Multiply both sides by } b. \\ ab - a^2 &= b^2 - a^2 & & \text{Subtract } a^2 \text{ from both sides.} \\ a(b-a) &= (b+a)(b-a) & & \text{Factor} \\ a &= b+a & & \text{Divide both sides by } b-a. \\ a &= 2a & & \text{Since } b+a = 2a. \\ 1 &= 2 & & \text{Divide both sides by } a. \end{aligned}$$

The erroneous step was dividing both sides by $b - a$, which was dividing by 0 since $a = b \Leftrightarrow b - a = 0$. There are other (equally invalid) ways to prove $1 = 2$.

3.6 What's wrong with this proof of 1 = -1?

You may also come across proofs of $1 = -1$, such as...

$$\begin{aligned} -1 &= -1 & & \\ \frac{-1}{1} &= \frac{1}{-1} & & \\ \sqrt{\frac{-1}{1}} &= \sqrt{\frac{1}{-1}} & & \text{Take the square root of both sides.} \\ \frac{\sqrt{-1}}{\sqrt{1}} &= \frac{\sqrt{1}}{\sqrt{-1}} & & \text{Since } \sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}. \\ \frac{i}{1} &= \frac{1}{i} & & \text{Since } i = \sqrt{-1}. \\ i^2 &= 1 & & \text{Cross-multiply} \\ -1 &= 1 & & \text{Since } i^2 = -1. \end{aligned}$$

The erroneous step was assuming $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$ always holds when a or b is negative. An argument can be made that while taking the square root of both sides, the "negative" root was not considered, else we would have concluded $-1 = -1$ to be valid and $-1 = 1$ to be invalid, i.e. extraneous. The aforementioned erroneous assumption is the *reason* $-1 = 1$ is extraneous.

3.7 Is $.\overline{99}$ equal to 1?

Discussions (sometimes quite colorful) regarding the equation $.\overline{99} = 1$ seem to pop up from time to time in certain mathematics newsgroups, including `alt.algebra.help`. These are some of the more common questions and answers.

(Q1) No matter how many decimal places one cares to expand it, it's always going to be less than 1. You tell me how many digits to carry it out, and I'll tell you what the difference is between that and 1. Doesn't this imply they are not equal?

(A1) If I tell you any *particular* decimal place to carry out the expansion (no matter how many digits that may be) then true, the result will always be less than 1. However, we wouldn't be talking about $.\overline{99}$ anymore, but rather something else entirely that has the form $.9999\ldots9$, where $\ldots$ could be any finite number of nines (regardless of how many that may be,) followed by a final terminating 9. This is not equivalent to $.\overline{99}$, which has no such terminating position. The pattern repeats indefinitely, which is the very meaning of the overbar in the expression. The decimal expansion *never* ends. Realize, if there is to be a "difference" between $.\overline{99}$ and 1 then we could equate $.\overline{99}$ to $.9999\ldots9$ by specifying an appropriate number of digits in $\ldots$. In other words, it would be a terminating decimal. This is a contradiction, since the expression $.\overline{99}$ by definition (of the overbar) is a *repeating* decimal.

(Q2) Could it be that a real number between $.\overline{99}$ and 1 does in fact exist, but we just don't know how to properly describe it with decimal notation?

(A2) If such a number exists, it could certainly be described to some extent as a decimal, since all real numbers have decimal expansions that either:

- terminate, e.g. $\frac{1}{2} = .5$, $\frac{12}{100} = .12$
- do not terminate but contain some pattern that repeats indefinitely, e.g. $\frac{7}{9} = .\overline{77}$, $\frac{9}{11} = .\overline{81}$
- neither terminate nor contain any repeating pattern, e.g. $\sqrt{2} = 1.4142135\ldots$, $\pi = 3.1415926\ldots$.

Such a number (between $.\overline{99}$ and 1) would clearly have a decimal expansion of the form $.999\ldots(\textit{something})$. First, if it terminated anywhere it would certainly be less than 1 (as desired) but it would also necessarily be less than $.\overline{99}$, since $.\overline{99}$ is greater than anything of the form $.999\ldots9$ for any number of digits in $\ldots$, as was explained in (A1). Therefore, such a number would not be *between* $.\overline{99}$ and 1. Second, if such a number did not terminate but repeated, then it would be of the form $.999\overline{999}$ which is, well, equal to $.\overline{99}$, hence not *between* $.\overline{99}$ and 1. Finally, if such a number neither terminated nor repeated, then some of the digits in the decimal expansion must be less than 9, meaning this number would be less than $.\overline{99}$, hence not *between* $.\overline{99}$ and 1. Since all three cases have been ruled out, we conclude that no such real number exists that is between $.\overline{99}$ and 1, therefore they *must* be the same number.

(Q3) Isn't the difference between $.\overline{99}$ and 1 something like $.0000\overline{000}001$?

(A3) That is not a valid expression for a real number. The repeating pattern $\overline{000}$ means you would never get to the terminating 1, so if this expression was any number at all—and I'm not saying it is—it would have to be 0 since $.000\overline{000} = 0$. Furthermore, if the difference between two real numbers is 0, then they are in fact the same number ($a = b \iff a - b = b - a = 0$.)

(Q4) I have seen some rather simple algebraic proofs that show $.\overline{99} = 1$. Are they valid, and if not where do they break down?

(A4) Although not formal proofs, they are indeed valid demonstrations. The assumptions they rely on can be proven, with appropriate methods, to be valid (including the assumptions made in **A1-A3**.)

$$
\begin{aligned}
x &= .\overline{99} \\
10x &= 10(.\overline{99}) \\
10x &= 9.\overline{99} \\
10x - x &= 9.\overline{99} - .\overline{99} \\
9x &= 9 \\
x &= 1
\end{aligned}
$$

One assumption made here is that it is indeed legal to move the decimal point over one position when multiplying such a repeating decimal by 10, just like you would a terminating decimal.

Another simple demonstration relies on the assumption that $\frac{1}{3}$ is equal to $.\overline{33}$. Many who are at first skeptical of $.\overline{99} = 1$ seem to have no problem understanding that $.\overline{33} = \frac{1}{3}$, perhaps because it is easily verified with long division:

$$
\begin{array}{r}
.333\ldots \\
3\overline{)1.000\ldots} \\
\underline{9} \\
10 \\
\underline{9} \\
10 \\
\underline{9} \\
1
\end{array}
\quad \text{and so on} \ldots
$$

So we have $.\overline{33} = \frac{1}{3}$. Multiplying both sides of this equation by 3 yields $.\overline{99} = 1$. But for that matter, we can simply divide 1 by itself and draw the same conclusion...

$$
\begin{array}{r}
.999\ldots \\
1\overline{)1.000\ldots} \\
\underline{9} \\
10 \\
\underline{9} \\
10 \\
\underline{9} \\
1
\end{array}
\quad \text{and so on} \ldots
$$

Don't be too quick to assume a logical fallacy here. Although it is certainly true that 1 goes into 1.0 *1.0* times, it is also true that 1 goes into 1.0 *.9 times with a remainder of .1*. Then, 1 goes into .10 *.09 times with a remainder of .01*, and so forth. Of course this is not the standard way long division is usually performed, but it is valid nonetheless.

(Q5) What's all this I hear about series? What does a series have to do with $.\overline{99}$?

(A5) Using an infinite sum (a.k.a. series) is a more appropriate method of **proving** $.\overline{99} = 1$. A series is basically a never ending addition problem, so $.\overline{99}$ can be represented by the series:

$$.9 + .09 + .009 + .0009 + \cdots$$

which is the same as ...

$$\frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \frac{9}{10000} + \cdots$$

This expression is a direct result of the very meaning of the base 10 number system that we all learned in school (think place value.) Examining this expression, we can easily see a pattern. Each numerator is 9 and each denominator is a successive power of 10 ($10^1, 10^2$, etc.). Using Sigma notation we have:

$$\sum_{n=1}^{\infty} \frac{9}{10^n}$$

Not all series actually have a sum. Those that do are said to converge while those that do not are said to diverge. Convergent and divergent series are defined as follows:

For the infinite series Σa_n, the $\mathbf{n^{th}}$ partial sum is given by:

$$S_n = a_1 + a_2 + \cdots + a_n$$

If the sequence of partial sums $\{S_n\}$ converges to **L**, then the series Σa_n **converges** and **L** is called the **sum of the series**. If $\{S_n\}$ diverges, then the series diverges.

In other words, the sum of an infinite series is equal to the limit of its $\mathbf{n^{th}}$ partial sum, if this limit exists (if it does not exist, the series has no sum.) Also, an infinite sequence converges to **L** if the limit of its $\mathbf{n^{th}}$ term is **L**. Furthermore, the sum of the series is also equal to **L**. Using the definition, we can directly prove $.\overline{99} = 1$. Let's take some partial sums:

$$\begin{aligned}
S_1 &= \frac{9}{10} \\
S_2 &= \left(\frac{9}{10} + \frac{9}{100}\right) = \frac{99}{100} \\
S_3 &= \left(\frac{9}{10} + \frac{9}{100} + \frac{9}{1000}\right) = \frac{999}{1000} \\
S_4 &= \left(\frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \frac{9}{10000}\right) = \frac{9999}{10000}
\end{aligned}$$

A pattern is easily recognizable that lets us generalize the $\mathbf{n^{th}}$ partial sum. The denominators of the partial sums are successive powers of 10. The numerators are all 1 less than the denominator, so we can write the $\mathbf{n^{th}}$ partial sum as:

$$S_n = \frac{10^n - 1}{10^n}$$

According to the definition, the series $\sum_{n=1}^{\infty} \frac{9}{10^n}$ will converge to $\lim_{n\to\infty} \frac{10^n - 1}{10^n}$, if the limit exists. The limit indeed exists (it's 1, proof omitted) and we conclude that:

$$.\overline{99} = \sum_{n=1}^{\infty} \frac{9}{10^n} = \lim_{n\to\infty} \frac{10^n - 1}{10^n} = 1$$

At this point, it is inevitable that someone will argue something to the effect of, "just because that limit is 1 doesn't necessarily mean the sum of the series is 1 because it is well known that things don't *have* to equal their limit value." But this line of reasoning is invalid, since the very definition of convergent series tells us this limit *is* the sum of the series.

Of course, we don't necessarily have to use the definition each and every time we need to find the sum of a series. There are various formulas that we can use to find the sums of certain *types* of series. In our case, you may recognize $\sum_{n=1}^{\infty} \frac{9}{10^n}$ as a **geometric** series with a first term a of $\frac{9}{10}$ and a ratio r of $\frac{1}{10}$. If $0 < |r| < 1$, a geometric series will converge. Furthermore, it will converge to the sum:

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} \qquad 0 < |r| < 1$$

Letting $a = \frac{9}{10}, r = \frac{1}{10}$ we have:

$$.\overline{99} = \sum_{n=0}^{\infty} \left(\frac{9}{10}\right)\left(\frac{1}{10}\right)^n = \frac{\frac{9}{10}}{1-\frac{1}{10}} = \frac{9}{10-1} = \frac{9}{9} = 1$$

Some Final Thoughts.

Since all repeating decimals are rational numbers, $.\overline{99}$ itself is a rational number. But which rational number is it? Is it $\frac{999}{1000}$? Is it $\frac{999999}{1000000}$? Is it $\frac{99999999999999}{100000000000000}$? It can be none of these, even if you carry the pattern out as far as you like, because they all have decimal representations that *terminate*. So, how can we easily convert not only $.\overline{99}$, but any repeating decimal into its fractional form? One commonly used method to accomplish this is to divide the part that repeats (the period) by the same number of nines as there are digits in the period. This will become clear with a few examples, that are easily verifiable with long division:

- $.\overline{33}$ has a period consisting of one digit. Dividing this one digit by one nine yields $\frac{3}{9}$ which simplifies to $\frac{1}{3}$.
- $.\overline{77}$ has a period consisting of one digit. Dividing this one digit by one nine yields $\frac{7}{9}$.
- $.\overline{81}$ has a period consisting of two digits. Dividing these two digits by two nines yields $\frac{81}{99}$ which simplifies to $\frac{9}{11}$.
- $.\overline{857142}$ has a period consisting of 6 digits. Dividing these 6 digits by 6 nines gives $\frac{857142}{999999}$ which simplifies to $\frac{6}{7}$.

And finally, $.\overline{99}$ has a period consisting of one digit. Dividing this one digit by one nine yields $\frac{9}{9}$ which simplifies to 1.

3.8 What is the order of operations?

Because mathematical expressions involving multiple operations could otherwise be evaluated to different values by different reasonable people, we have an agreement on the specific order in which operations are to be performed. This assures we will all get the same result when evaluating the same expression, assuming we are using the same agreement.

Although alternative agreements do exist (some programming languages and software packages are examples,) it is by and large the following agreement that is most commonly used in a classroom setting and thus the usual convention on `alt.algebra.help`:

1. Simplify inside grouping symbols, working from the innermost to the outermost. Grouping symbols include parentheses, brackets, and the fraction bar.
2. Simplify powers/roots.
3. Perform multiplication/division as they occur from left to right.
4. Perform addition/subtraction as they occur from left to right.

Examples:

$$\begin{aligned} 6 \div 3 + 2^3 \cdot 5 &= 6 \div 3 + \mathbf{8} \cdot 5 \\ &= \mathbf{2} + 8 \cdot 5 \\ &= 2 + \mathbf{40} \\ &= 42 \end{aligned}$$

$$\begin{aligned} (8+6) \div 7 \cdot 3 - 6 &= \mathbf{14} \div 7 \cdot 3 - 6 \\ &= \mathbf{2} \cdot 3 - 6 \\ &= \mathbf{6} - 6 \\ &= 0 \end{aligned}$$

$$\begin{aligned} \frac{-(-3)^3 + (-5)}{2(-8) - 5(3)} &= \frac{-(-\mathbf{27}) + (-5)}{2(-8) - 5(3)} \\ &= \frac{\mathbf{27 - 5}}{\mathbf{-16 - 15}} \\ &= \frac{22}{-31} \\ &= -\frac{22}{31} \end{aligned}$$

3.9 Can you help me understand word problems?

A good discussion of word problems and how to translate them into algebra can be found at the following sites:

- The Ask Dr. Math FAQ (http://mathforum.org/dr.math/faq/faq.word.problems.html).
- The Ask Dr. Math Archives (http://mathforum.org/dr.math/tocs/wordproblem.middle.html) also contains a long list of specific word problems. If you are having trouble with a specific word problem, this is well worth a look. Chances are, a problem very similar to yours will be listed along with the method of solution.
- Purplemath - Your Algebra Resource (http://purplemath.com/modules/translat.htm). Contains explanations of several specific types of word problems.

3.10 Can you help me understand "grazing animal" problems?

A good explanation of these and other "classic" problems can be found in the Ask Dr. Math FAQ:

- http://mathforum.org/dr.math/faq/faq.grazing.html

3.11 How can I find a square root without a calculator?

A good explanation of this can be found in the Ask Dr. Math FAQ:

- http://mathforum.org/dr.math/faq/faq.sqrt.by.hand.html

3.12 Given any date in any year, what day of the week is it on?

This and other interesting "calendar" information can be found in the Ask Dr. Math FAQ:

- http://mathforum.org/dr.math/faq/faq.calendar.html

3.13 Why is division by 0 undefined?

A good explanation of this can be found in the Ask Dr. Math FAQ:

- http://mathforum.org/dr.math/faq/faq.divideby0.html

3.14 Where does the Quadratic Formula come from?

The method of completing the square can be used to solve *any* quadratic equation, i.e. any equation that can be put in the form $ax^2 + bx + c = 0$ where $a \neq 0$. This includes the very equation $ax^2 + bx + c = 0$ itself...

$$ax^2 + bx + c = 0$$

$$ax^2 + bx = -c$$

Get the constant term, c, alone on one side.

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

Since the coefficient of x^2 needs to be 1,

divide both sides of the equation by a.

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}$$

"Complete the square" by squaring half the coefficient of x

and adding the result to both sides. Note that

$\frac{1}{2} \cdot \frac{b}{a} = \frac{b}{2a}$ and $\left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2}$.

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$$

Factor the perfect square trinomial.

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

Rewrite the right hand side as a single fraction.

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

Take the square root of both sides.

$$x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}$$

Since $4a^2 = (2a)^2$.

$$x = \frac{\pm\sqrt{b^2 - 4ac}}{2a} - \frac{b}{2a}$$

Subtract $\frac{b}{2a}$ from both sides.

$$x = \frac{\pm\sqrt{b^2 - 4ac} - b}{2a}$$

Rewrite the right hand side as a single fraction.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Rearrange the numerator.

The Quadratic Formula: The solutions of $ax^2 + bx + c = 0$, where $a \neq 0$, are:

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Chapter 4

Reference Center

4.1 Algebra

4.1.1 Arithmetic Operations

For real numbers a, b, c, d such that no denominator is 0:

$$ab + ac = a(b + c) \qquad \frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd} \qquad \frac{a + b}{c} = \frac{a}{c} + \frac{b}{c}$$

$$\frac{\left(\frac{a}{b}\right)}{\left(\frac{c}{d}\right)} = \left(\frac{a}{b}\right)\left(\frac{d}{c}\right) = \frac{ad}{bc} \qquad \frac{\left(\frac{a}{b}\right)}{c} = \frac{a}{bc} \qquad \frac{a}{\left(\frac{b}{c}\right)} = \frac{ac}{b}$$

$$a\left(\frac{b}{c}\right) = \frac{ab}{c} \qquad \frac{a - b}{c - d} = \frac{b - a}{d - c} \qquad \frac{ab + ac}{a} = b + c$$

4.1.2 Absolute Value

For real numbers a, b and positive real number k :

$$|a| \equiv \begin{cases} a, & \text{if } a \geq 0 \\ -a, & \text{if } a < 0 \end{cases} \qquad \text{Alternatively,} \qquad |a| \equiv \sqrt{a^2}$$

$$|a| \geq 0 \qquad |-a| = |a| \qquad |a| \cdot |b| = |ab|$$

$$\left|\frac{a}{b}\right| = \frac{|a|}{|b|}, \quad b \neq 0 \qquad |a + b| \leq |a| + |b| \qquad |a^n| = |a|^n$$

$$-|a| \leq a \leq |a| \qquad \underbrace{|a| \leq k \iff -k \leq a \leq k}_{\text{also true if } \leq \text{ is replaced by } <} \qquad \underbrace{k \leq |a| \iff k \leq a \text{ or } a \leq -k}_{\text{also true if } \leq \text{ is replaced by } <}$$

$$|a| = b \iff a = b \text{ or } a = -b$$

4.1.3 Exponents

For nonzero real numbers a, b and integers x, y:

$$a^0 = 1 \qquad (ab)^x = a^x b^x \qquad a^x a^y = a^{x+y}$$

$$\frac{a^x}{a^y} = a^{x-y} \qquad \left(\frac{a}{b}\right)^x = \frac{a^x}{b^x} \qquad a^{-x} = \frac{1}{a^x}$$

4.1.4 Radicals

For positive integers m, n and real numbers a, b such that $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real:

$$\sqrt{a} = a^{1/2} \qquad \sqrt[n]{a} = a^{1/n} \qquad a^{m/n} = (a^{1/n})^m = (a^m)^{1/n} = \sqrt[n]{a^m}$$

$$\sqrt[n]{ab} = \left(\sqrt[n]{a}\right)\left(\sqrt[n]{b}\right) \qquad \sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}, b \neq 0 \qquad \sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$$

$$(\sqrt[n]{a})^n = a \qquad \sqrt[n]{a^n} = \begin{cases} |a|, & \text{for } n \text{ even} \\ a, & \text{for } n \text{ odd} \end{cases}$$

4.1.5 Factoring

Difference of two squares:	$x^2 - y^2 = (x+y)(x-y)$
Difference/sum of two cubes:	$x^3 \pm y^3 = (x \pm y)(x^2 \mp xy + y^2)$
Difference of two fourth powers:	$x^4 - y^4 = (x^2 - y^2)(x^2 + y^2) = (x+y)(x-y)(x^2+y^2)$
Perfect square trinomial:	$x^2 \pm 2xy + y^2 = (x \pm y)^2$
Factoring by Grouping:	$acx^3 + adx^2 + bcx + bd = ax^2(cx+d) + b(cx+d) = (ax^2+b)(cx+d)$

4.1.6 Factors and Zeros of Polynomials

Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial. If $p(a) = 0$, then a is a *zero* of the polynomial and a solution of the equation $p(x) = 0$. Furthermore, $(x - a)$ is a *factor* of the polynomial.

4.1.7 Fundamental Theorem of Algebra

An nth degree polynomial has n (not necessarily distinct) zeros. Although all of these zeros may be imaginary, a real polynomial of odd degree must have at least one real zero.

4.1.8 Rational Zero Theorem

If $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ has integer coefficients, then every *rational zero* of p is of the form $x = \frac{r}{s}$, where r is a factor of a_0 and s is a factor of a_n.

4.1.9 Zero Factor Property

If a and b are complex numbers, with $ab = 0$, then $a = 0$ or $b = 0$ or both.

4.1.10 Quadratic Formula

If $p(x) = ax^2 + bx + c$, $a \neq 0$, then the zeros of p are $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$.

4.1.11 Binomial Theorem

$(x+y)^2 = x^2 + 2x + y^2$

$(x-y)^2 = x^2 - 2xy + y^2$

$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$

$(x-y)^3 = x^3 - 3x^2y + 3xy^2 - y^3$

$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$

$(x-y)^4 = x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$

$$(x+y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2 + \cdots + nxy^{n-1} + y^n$$

$$(x-y)^n = x^n - nx^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2 - \cdots \pm nxy^{n-1} \mp y^n$$

4.1.12 Distance Formula

The distance between $(x_1\,, y_1)$ and $(x_2\,, y_2)$ is:

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

4.1.13 Midpoint Formula

The midpoint of the line segment with endpoints $(x_1\,, y_1)$ and $(x_2\,, y_2)$ is:

$$\left(\frac{x_1 + x_2}{2}\,,\ \frac{y_1 + y_2}{2}\right).$$

4.1.14 Slope

The slope m of the line through $(x_1\,, y_1)$ and $(x_2\,, y_2)$ is:

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}\,,\quad \Delta x \neq 0.$$

4.1.15 Equations of Lines

- **Point-Slope Form:** If a line has slope m and passes through the point $(x_1\,, y_1,)$ then an equation of the line is $y - y_1 = m(x - x_1)$.
- **Slope-Intercept Form:** If a line has slope m and y-intercept $(0\,, b)$, then an equation of the line is $y = mx + b$.

- **Standard Form:** Any line has an equation of the form $Ax + By = C$, where A, B, C are real and A and B are not both 0. A, B, C should share no common factors, e.g. $x + 2y = 4$ is preferable to $2x + 4y = 8$. Additionally, some prefer A to be nonnegative.
- **Vertical Line:** An equation of the vertical line passing through $(a\,,\,b)$ is $x = a$. Vertical lines have undefined slope.
- **Horizontal Line:** An equation of the horizontal line passing through $(a\,,\,b)$ is $y = b$. Horizontal lines have slope 0.
- **Parallel Lines:** Two distinct nonvertical lines are parallel if and only if they have the same slope.
- **Perpendicular Lines:** Two lines, neither of which are vertical, are perpendicular if and only if their slopes have a product of -1, i.e. if a line has slope m, then a perpendicular line has slope $-1/m$.

4.1.16 Functions

A **relation** between two sets X and Y is a set of ordered pairs $(x\,,\,y)$, where x is an element of X and y is an element of Y.

Let X and Y be sets of real numbers. A **real-valued function** f **of a real variable** x from X to Y is a relation that assigns to each number x in X exactly one number y in Y. The **domain** of f is the set X. The **codomain** of f is the set Y. The number y is the **image** of x under f and is denoted $f(x)$. The **range** of f is a subset of Y and consists of all images of numbers in X. The variable x is the **independant variable**, and the variable y is the **dependant variable**.

A function is **one-to-one** if to each y-value in the range there corresponds exactly one x-value in the domain. f is one-to-one if $a \neq b \implies f(a) \neq f(b)$.

Function Composition

Let f and g be functions. The function given by $(f \circ g)(x) = f(g(x))$ is called the **composite** of f with g. The domain of $f \circ g$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f.

Even/Odd Functions and Symmetry

A function f is **even** if $f(-x) = f(x)$. Even functions are symmetric about the y-axis. A function f is **odd** if $f(-x) = -f(x)$. Odd functions are symmetric about the origin.

Inverse Functions

- A function g is the **inverse** of the function f if $f(g(x)) = x$ for each x in the domain of g, and $g(f(x)) = x$ for each x in the domain of f. The function g is denoted f^{-1}.
- If g is the inverse of f, then f is the inverse of g.
- The domain of f^{-1} is equal to the range of f, and the range of f^{-1} is equal to the domain of f.
- The graph of f contains the point $(a\,,\,b)$ if and only if the graph of f^{-1} contains the point $(b\,,\,a)$. Inverse functions are symmetric about the line $y = x$.
- A function possesses an inverse if and only if it is one-to-one.

Finding the Inverse of a Function:

- Determine whether the function $y = f(x)$ has an inverse. It has an inverse if and only if it is one-to-one.
- Solve for x as a function of y.
- Interchange x and y. The result is $y = f^{-1}(x)$.
- Define the domain of f^{-1} to be the range of f.
- Verify that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.

4.1.17 Logarithms

$$y = \log_a x \iff x = a^y \qquad \log_a xy = \log_a x + \log_a y \qquad \log_a \frac{x}{y} = \log_a x - \log_a y$$

$$\log_a x^r = r \log_a x \qquad \log_a a = 1 \qquad \log_a 1 = 0$$

4.1.18 Combinatorics

- The number of arrangements of n things taken r at a time is $P(n, r) = \frac{n!}{(n-r)!}$.
- The number of ways to choose r elements from a group of n elements is $\binom{n}{r} = \frac{n!}{(n-r)!r!}$.

4.1.19 Probability

For any events E and F :

- $0 \leq P(E) \leq 1$
- $P(\text{certain event}) = 1$
- $P(\text{impossible event}) = 0$
- $P(E') = 1 - P(E)$
- $P(E \,\text{or}\, F) = P(E \cup F) = P(E) + P(F) - P(E \cap F)$.

4.2 Geometry

Some formulas involving common planar figures and solids can be found here:

- http://www.doe.virginia.gov/VDOE/Assessment/Manipulatives/GEOMETRY_BP2003.pdf

4.3 Trigonometry

4.3.1 Angles and Angle Measure

A geometric figure consisting of two rays with a common endpoint is called an **angle.** The common endpoint is called the **vertex** of the angle, and the rays are called the **sides** of the angle.

In the study of trigonometry, it is useful to generalize the concept of angle by letting the vertex coincide with the origin of the xy plane, and begin by letting both sides coincide with the positive x-axis. One side of the angle, called the **initial side**, remains fixed on the positive x-axis. The other side of the angle, called the **terminal side**, is then "rotated" until it reaches its terminal position. Such an angle is said to be in **standard position.**

A positive angle corresponds to a counterclockwise rotation, while a negative angle corresponds to a clockwise rotation. Generalizing the angle concept in this manner, we can consider angles of any size, positive *or* negative, since the rotation of the terminal side can exceed a complete revolution (i.e. exceed 360°,) and in either direction. In short, an angle in standard position may be considered with any desired real number measure.

Two angles in standard position in the same coordinate system are called **coterminal** angles if their terminal sides coincide. An angle in standard position with terminal side coinciding with a coordinate axis, is called a **quadrantal** angle.

A **central angle** of a circle is an angle with a vertex that coincides with the center of the circle, and sides that correspond to radii of the circle. The arc formed by the intersection of these sides and the circle itself, is said to **subtend** the angle, and the angle is said to be **subtended by** the arc.

Degree Measure

A central angle that is subtended by an arc equal in length to 1/360 of the circle's circumference, is said to have a measure of **one degree,** denoted 1°. For a circle with circumference C units, central angle of θ degrees subtended by an arc of s units, this relationship can be expressed as the proportion

$$\frac{\theta^\circ}{360^\circ} = \frac{s}{C},$$

θ° in decimal degrees; s and C in same units.

Thus if any two of θ, s, or C are known, the third can be found with simple algebra.

A degree is often further divided into **minutes** (1/60 degree) and **seconds** (1/60 minute,) or as decimal degrees.

A 90° angle is called a **right angle.** A 180° angle is called a **straight angle.** Angles less than 90° are said to be **acute,** while angles greater than 90° but less than 180° are said to be **obtuse.**

Radian Measure

A central angle of a circle that is subtended by an arc equal in length to the circle's radius, is said to have a measure of **one radian,** denoted 1 rad.

It follows that the radian measure of a central angle θ subtended by an arc of length s can be found by determining how many times the length of the radius r is contained in the arc length s. This suggests the formulas (s, r in the same units, θ in radians:)

$$\theta = \frac{s}{r}$$

$$s = r\theta.$$

Note: If the units of an angle's measure are not specified, in most contexts **radians are the default.**

Conversion Formulas

Since the circumference of a circle is $2\pi r$, the radian measure corresponding to $360°$ is easily found with the preceding formula

$$\theta = \frac{s}{r} \implies \theta = \frac{2\pi r}{r} \implies \theta = 2\pi.$$

Notice that the radius r cancels (and hence the units in which it is measured,) leaving the radian measure of an angle as a "unitless" number.

It follows that an angle with degree measure $180°$ corresponds to a radian measure of π. Thus, we have the proportion:

$$\frac{\theta°}{180°} = \frac{\theta\,\text{rad}}{\pi\,\text{rad}}$$

from which follow the conversion formulas:

$$\text{radians} = \text{degrees}\left(\frac{\pi}{180}\right)$$

$$\text{degrees} = \text{radians}\left(\frac{180}{\pi}\right).$$

4.3.2 Trigonometric Functions

Circular Definitions

Trigonometric functions (also called **circular functions**) are defined using the unit circle, $x^2 + y^2 = 1$. Let θ be a central angle, measured counterclockwise, subtended by the arc with initial point $(1, 0)$ and endpoint (x, y). The sine of θ, denoted $\sin\theta$, is defined as the vertical component of the endpoint of the arc. The cosine of θ, denoted $\cos\theta$, is defined as the horizontal component. The coordinates (x, y) of any point on the unit circle therefore correspond to $(\cos\theta, \sin\theta)$.

The ratio $\sin\theta/\cos\theta$ is defined as the tangent of θ, denoted $\tan\theta$.

The reciprocal of the sine of θ is defined as the cosecant of θ, denoted $\csc\theta$. The reciprocal of the cosine of θ is defined as the secant of θ, denoted $\sec\theta$. The reciprocal of the tangent of θ is defined as the cotangent of θ, denoted $\cot\theta$.

An obvious consequence of these "circular" definitions is the trigonometric functions are periodic with period 2π. That is,

$$\text{func}(2\pi n + \theta) = \text{func}(\theta),$$

where n is an integer and func is a trigonometric function. Additionally, The tangent and cotangent functions have period π.

The Sign of a Trigonometric Function

The sign of a trigonometric functional value of an angle θ (not quadrantal) can be easily determined from the quadrant in which the terminal side of θ lies, as summarized in the following table. The table considers only one revolution of the terminal side of θ around the unit circle (i.e. from 0 to 2π radians.) The same applies to integer multiples of 2π of these angles, since those will be coterminal with these.

θ	Quadrant	$\sin\theta$	$\csc\theta$	$\cos\theta$	$\sec\theta$	$\tan\theta$	$\cot\theta$
$0 < \theta < \frac{\pi}{2}$	I	positive	positive	positive	positive	positive	positive
$\frac{\pi}{2} < \theta < \pi$	II	positive	positive	negative	negative	negative	negative
$\pi < \theta < \frac{3\pi}{2}$	III	negative	negative	negative	negative	positive	positive
$\frac{3\pi}{2} < \theta < 2\pi$	IV	negative	negative	positive	positive	negative	negative

4.3.3 Right Triangle Ratios

Consider an angle θ between 0 and $\pi/2$ (0° and 90°) in standard position, and a perpendicular segment extending from the terminal side of θ to the x-axis. For now, we consider the perpendicular extending from the intersection of the terminal side of θ and the unit circle, i.e. the perpendicular extends from $(\cos\theta, \sin\theta)$ to the x-axis.

Notice that the figure formed by the terminal side of θ, the perpendicular segment, and the x-axis, is a **right triangle.** As a result of the circular definitions given previously, the vertical leg of this right triangle (which is the side **opposite** θ,) has length $\sin\theta$. The horizontal leg (which is the side **adjacent** θ,) has length $\cos\theta$. The hypotenuse has length 1, since it corresponds to a radius of the unit circle. By the Pythagorean Theorem, we immediately see that $\sin^2 x + \cos^2 x = 1$, which is a commonly used identity. More on identities later.

Consider the ratios of sides of this triangle:

$$\frac{\text{opposite}}{\text{hypotenuse}} = \frac{\sin\theta}{1} = \sin\theta \qquad \frac{\text{hypotenuse}}{\text{opposite}} = \frac{1}{\sin\theta} = \csc\theta$$

$$\frac{\text{adjacent}}{\text{hypotenuse}} = \frac{\cos\theta}{1} = \cos\theta \qquad \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{1}{\cos\theta} = \sec\theta$$

$$\frac{\text{opposite}}{\text{adjacent}} = \frac{\sin\theta}{\cos\theta} = \tan\theta \qquad \frac{\text{adjacent}}{\text{opposite}} = \frac{\cos\theta}{\sin\theta} = \cot\theta$$

Since similar triangles have corresponding sides that are in the same proportion, the values of these ratios are equal for a given acute angle θ **regardless of the right triangle it appears in.** In other words, $\sin\theta = \frac{\text{opposite}}{1}$ for this particular right triangle with hypotenuse 1, but in general $\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$ for **any** right triangle, and likewise for the other ratios.

So for $0 < \theta < \frac{\pi}{2}$ we define the trigonometric ratios of a right triangle:

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \csc\theta = \frac{\text{hypotenuse}}{\text{opposite}}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} \qquad \cot\theta = \frac{\text{adjacent}}{\text{opposite}}$$

4.3.4 Reference Triangles, Reference Angles

Now we extend our "right triangle" definitions of the trig functions to have a domain of *all real numbers,* not just numbers in the interval $0 < \theta < \pi/2$.

Consider again our right triangle in the first quadrant, with hypotenuse corresponding to the terminal side of θ and adjacent side corresponding to the x-axis. There exists a similar triangle in QII, also with hypotenuse 1 and sharing a side with the x-axis. There exists similar triangles in QIII and QIV also, all with hypotenuse 1, and all sharing a side with the x-axis.

Because of the properties of similar triangles, each of these triangles has an acute angle α (always taken positive) equal to our original θ, and all six trigonometric ratios of any of these similar triangles are equal to the corresponding ratios of our original triangle in QI. For that matter, the hypotenuse of these triangles need not be 1 and the same holds true, again due to properties of similar triangles.

With respect to an angle θ in standard position, such a triangle is called a **reference triangle** (or related triangle.) The angle α (always taken positive) between the terminal side of θ and the x-axis, is called a **reference angle** (or related angle.)

For example, when $\theta = 2\pi/3$ (120 degrees), the reference angle is $\pi/3$ (60 degrees.)

Since the hypotenuse of such a triangle corresponds to the terminal side of an angle in standard position, every real number angle has such a reference triangle and reference angle (except for quadrantal angles which will be considered later.) We may use any reference triangle to consider a trigonometric ratio of a given angle, just as we do for the reference triangle in the first quadrant. When so doing, however, we have to **assign the correct sign (plus or minus)** to the ratio's value, as discussed previously, depending on which quadrant the terminal side of θ lies in. To continue with our example, since $2\pi/3$ has reference angle $\pi/3$, $\sin(2\pi/3) = \sin(\pi/3)$, while $\cos(2\pi/3) = -\cos(\pi/3)$, since sine is positive in both QI and QII, while cosine is positive in QI but negative in QII.

In closing our description of reference triangles, it is worth mentioning that although the hypotenuse of such a triangle need not be 1, it is usually convenient to **let the hypotenuse be 1** when considering a reference triangle/angle, since this simplifies matters.

4.3.5 Special Angles, Exact Values

Generally speaking, the exact value of a trigonometric function of a given angle cannot be found easily, if at all. In many applications an approximation will suffice, however, there are certain **special angles** such that the exact value of any trigonometric function can be easily found and expressed.

Quadrantal Angles

These angles have terminal side coinciding with a coordinate axis, thus a trigonometric functional value of such an angle is determined by the coordinates of the point where the terminal side (axis) intersects the unit circle. Recall that, by definition, the point (x, y) on the unit circle corresponds to $(\cos\theta\,,\,\sin\theta)$.

θ	Point	$\sin\theta$	$\csc\theta$	$\cos\theta$	$\sec\theta$	$\tan\theta$	$\cot\theta$
0, or $0°$	$(1\,,\,0)$	0	not defined	1	1	0	not defined
$\frac{\pi}{2}$, or $90°$	$(0\,,\,1)$	1	1	0	not defined	not defined	0
π, or $180°$	$(-1\,,\,0)$	0	not defined	-1	-1	0	not defined
$\frac{3\pi}{2}$, or $270°$	$(0\,,\,-1)$	-1	-1	0	not defined	not defined	0

Acute Angles in "Special" Right Triangles

Recall that in a $30° - 60° - 90°$ triangle, the length of the hypotenuse is twice that of the side opposite the 30 degree angle, and the side adjacent to the 30 degree angle is $\sqrt{3}$ times the side opposite the 30 degree angle.

In a $45° - 45° - 90°$ triangle, the two legs are of equal length, and the hypotenuse is $\sqrt{2}$ times that length.

These relationships, when applied to the right triangle definitions of the trig functions, allow us to easily find and express the exact value of any trig function of such an angle.

The following table summarizes these special values for the special angles in QI (i.e. between 0 and $\pi/2$ radians) so the same applies to integer multiples of 2π of these angles. For special angles in QII through QIV, simply determine the desired trig value of the **reference angle,** then assign the **correct sign (plus or minus)** according to the quadrant in which the terminal side lies.

A consequence of these special angle relationships is, given one trigonometric functional value of such an angle, the other five are easily determined from the right triangle definitions and the Pythagorean Theorem.

θ	$\sin\theta$	$\csc\theta$	$\cos\theta$	$\sec\theta$	$\tan\theta$	$\cot\theta$
$\frac{\pi}{6}$, or 30°	$1/2$	2	$\sqrt{3}\,/\,2$	$(2\sqrt{3})\,/\,3$	$\sqrt{3}\,/\,3$	$\sqrt{3}$
$\frac{\pi}{4}$, or 45°	$(\sqrt{2})\,/\,2$	$\sqrt{2}$	$(\sqrt{2})\,/\,2$	$\sqrt{2}$	1	1
$\frac{\pi}{3}$, or 60°	$\sqrt{3}\,/\,2$	$(2\sqrt{3})\,/\,3$	$1/2$	2	$\sqrt{3}$	$\sqrt{3}\,/\,3$

4.3.6 Inverse Trigonometric Functions

By restricting the domains of the trigonometric functions, inverse trigonometric functions are defined as:

			domain	range
$y = \arcsin x$	$\iff$	$x = \sin y$	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$y = \arccos x$	$\iff$	$x = \cos y$	$[-1, 1]$	$[0, \pi]$
$y = \arctan x$	$\iff$	$x = \tan y$	$(-\infty, \infty)$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
$y = \operatorname{arccot} x$	$\iff$	$x = \cot y$	$(-\infty, \infty)^1$	$(0, \pi)^1$
$y = \operatorname{arcsec} x$	$\iff$	$x = \sec y$	$(-\infty, -1] \cup [1, \infty)^1$	$\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]^1$
$y = \operatorname{arccsc} x$	$\iff$	$x = \csc y$	$(-\infty, -1] \cup [1, \infty)^1$	$\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]^1$

There is an alternative notation which uses what *appears* to be, but is not, an exponent of -1. For example, using this notation $\sin^{-1} = \arcsin$, *not* csc.

[1]Other conventions for the domain and range exist.

4.3.7 Law Of Sines

For a triangle with angles α, β, γ and respective opposite sides a, b, c :

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$$

4.3.8 Law Of Cosines

The Pythagorean Theorem for right triangles can be generalized to apply to oblique triangles as well. For a triangle with angles $\alpha,\ \beta,\ \gamma$ and respective opposite sides $a,\ b,\ c$:

$$a^2 = b^2 + c^2 - 2bc\,\cos\alpha$$

$$b^2 = a^2 + c^2 - 2ac\,\cos\beta$$

$$c^2 = a^2 + b^2 - 2ab\,\cos\gamma$$

4.3.9 Identities

Reciprocal Identities

$$\sin x = \frac{1}{\csc x} \qquad \sec x = \frac{1}{\cos x} \qquad \tan x = \frac{1}{\cot x}$$

$$\csc x = \frac{1}{\sin x} \qquad \cos x = \frac{1}{\sec x} \qquad \cot x = \frac{1}{\tan x}$$

Quotient Identities

$$\tan x = \frac{\sin x}{\cos x} \qquad \cot x = \frac{\cos x}{\sin x}$$

Pythagorean Identities

$$\sin^2 x + \cos^2 x = 1 \qquad 1 + \tan^2 x = \sec^2 x \qquad 1 + \cot^2 x = \csc^2 x$$

Cofunction Identities

$$\sin\left(\frac{\pi}{2} - x\right) = \cos x \qquad \cos\left(\frac{\pi}{2} - x\right) = \sin x$$

$$\csc\left(\frac{\pi}{2} - x\right) = \sec x \qquad \tan\left(\frac{\pi}{2} - x\right) = \cot x$$

$$\sec\left(\frac{\pi}{2} - x\right) = \csc x \qquad \cot\left(\frac{\pi}{2} - x\right) = \tan x$$

Identities For Negatives

$$\sin(-x) = -\sin x \qquad \cos(-x) = \cos x$$

$$\csc(-x) = -\csc x \qquad \tan(-x) = -\tan x$$

$$\sec(-x) = \sec x \qquad \cot(-x) = -\cot x$$

Sum and Difference Identities

$$\sin(x \pm y) = \sin x \, \cos y \pm \cos x \, \sin y$$

$$\cos(x \pm y) = \cos x \, \cos y \mp \sin x \, \sin y$$

$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \, \tan y}$$

Double-Angle Identities

$$\sin 2x = 2 \sin x \, \cos x \qquad \cos 2x = \begin{cases} \cos^2 x - \sin^2 x \\ 1 - 2\sin^2 x \\ 2\cos^2 x - 1 \end{cases} \qquad \tan 2x = \begin{cases} \dfrac{2\tan x}{1 - \tan^2 x} \\ \dfrac{2\cot x}{\cot^2 x - 1} \\ \dfrac{2}{\cot x - \tan x} \end{cases}$$

Half-Angle Identities

$$\left.\begin{aligned} \sin \frac{x}{2} &= \pm\sqrt{\frac{1-\cos x}{2}} \\ \cos \frac{x}{2} &= \pm\sqrt{\frac{1+\cos x}{2}} \\ \tan \frac{x}{2} &= \frac{1-\cos x}{\sin x} = \frac{\sin x}{1+\cos x} = \pm\sqrt{\frac{1-\cos x}{1+\cos x}} \end{aligned}\right\} \text{Sign is determined by quadrant in which } \frac{x}{2} \text{ lies.}$$

$$\sin^2 x = \frac{1-\cos 2x}{2} \qquad \cos^2 x = \frac{1+\cos 2x}{2} \qquad \tan^2 x = \frac{1-\cos 2x}{1+\cos 2x}$$

Product-To-Sum Identities

$$\sin x\,\cos y = \frac{1}{2}\big[\sin(x+y)+\sin(x-y)\big]$$

$$\cos x\,\sin y = \frac{1}{2}\big[\sin(x+y)-\sin(x-y)\big]$$

$$\sin x\,\sin y = \frac{1}{2}\big[\cos(x-y)-\cos(x+y)\big]$$

$$\cos x\,\cos y = \frac{1}{2}\big[\cos(x+y)+\cos(x-y)\big]$$

Sum-To-Product Identities

$$\sin x+\sin y = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

$$\sin x-\sin y = 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

$$\cos x+\cos y = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

$$\cos x-\cos y = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

4.4 Calculus

4.4.1 Limits and Continuity

Definition of a limit Let f be a function defined on an open interval containing c (except possibly at c) and let L be a real number. The statement

$$\lim_{x\to c} f(x) = L$$

means that for each $\epsilon > 0$ there exists a $\delta > 0$ such that if

$$0 < |x-c| < \delta, \quad \text{then} \quad |f(x)-L| < \epsilon.$$

Definition of an Infinite Limit Let f be a function defined on an open interval containing c (except possibly at c.) The statement

$$\lim_{x \to c} f(x) = \infty$$

means that for each $M > 0$ there exists a $\delta > 0$ such that

$$f(x) > M \quad \text{whenever} \quad 0 < |x - c| < \delta.$$

Similarly, the statement

$$\lim_{x \to c} f(x) = -\infty$$

means that for each $N < 0$ there exists a $\delta > 0$ such that

$$f(x) < N \quad \text{whenever} \quad 0 < |x - c| < \delta.$$

To define the **infinite limit from the left,** replace $0 < |x - c| < \delta$ by $c - \delta < x < c$. To define the **infinite limit from the right,** replace $0 < |x - c| < \delta$ by $c < x < c + \delta$.

Definition of Limits at Infinity Let L be a real number. The statement

$$\lim_{x \to \infty} f(x) = L$$

means that for each $\epsilon > 0$ there exists an $M > 0$ such that

$$|f(x) - L| < \epsilon \quad \text{whenever} \quad x > M.$$

The statement

$$\lim_{x \to -\infty} f(x) = L$$

means that for each $\epsilon > 0$ there exists an $N < 0$ such that

$$|f(x) - L| < \epsilon \quad \text{whenever} \quad x < N.$$

Properties Of Limits Let b and c be real numbers, let n be a positive integer, and let f and g be functions with the following limits:

$$\lim_{x \to c} f(x) = L \quad \text{and} \quad \lim_{x \to c} g(x) = K$$

Constant: $\lim_{x \to c} b = b$

Scalar multiple: $\lim_{x \to c} [b\, f(x)] = bL$

Sum or difference: $\lim_{x \to c} [f(x) \pm g(x)] = L \pm K$

Product: $\lim_{x \to c} [f(x) g(x)] = LK$

Quotient: $\lim_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{K}, \quad K \neq 0$

Power: $\lim_{x \to c} [f(x)]^n = L^n$

Functions That Agree at All But One Point Let c be a real number and let $f(x) = g(x)$ for all $x \neq c$ in an open interval containing c. If $\lim_{x\to c} g(x)$ exists, then $\lim_{x\to c} f(x)$ also exists and is equal to $\lim_{x\to c} g(x)$.

The Squeeze Theorem If $h(x) \leq f(x) \leq g(x)$ for all x in an open interval containing c, except possibly at c, and if

$$\lim_{x\to c} h(x) = L = \lim_{x\to c} g(x)$$

then $\lim_{x\to c} f(x)$ exists and is equal to L.

Two Special Trigonometric Limits

$$\lim_{x\to 0} \frac{\sin x}{x} = 1 \qquad \lim_{x\to 0} \frac{1-\cos x}{x} = 0$$

Continuity A function f is **continuous at c** if the following three conditions are met:

$f(c)$ is defined.

$\lim_{x\to c} f(x)$ exists.

$\lim_{x\to c} f(x) = f(c)$.

A function f is **continuous on an open interval** (a, b) if it is continuous at each point in the interval. A function f is **continuous on a closed interval** $[a, b]$ if it is continuous on the open interval (a, b) and

$$\lim_{x\to a^+} f(x) = f(a) \quad \text{and} \quad \lim_{x\to b^-} = f(b).$$

4.4.2 Differentiation

The Tangent Line

If f is defined on an open interval containing c, and if the limit

$$\lim_{\Delta x\to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x\to 0} \frac{f(c+\Delta x) - f(c)}{\Delta x} = m$$

exists, then the line passing through $(c\,,\, f(c))$ with slope m is the **tangent line** to the graph of f at the point $(c\,,\, f(c))$.

The Derivative

The **derivative** of f at x, denoted $f'(x)$, is defined as

$$f'(x) \equiv \lim_{\Delta x\to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Notation for the Derivative

$$f'(x) \qquad \frac{dy}{dx} \qquad y' \qquad \frac{d}{dx}[f(x)] \qquad D_x[y]$$

Extrema

Critical Number If f is defined at c and either $f'(c) = 0$ or $f'(c)$ is undefined, then c is a **critical number** of f.

Extreme Values

- Assume f is defined on an open interval I containing c. If $f(c) \leq f(x)$ for all x in I, then $f(c)$ is the **minimum** of f on I. If $f(c) \geq f(x)$ for all x in I, then $f(c)$ is the **maximum** of f on I.
- The minimum/maximum values are the **extreme values,** or **extrema,** of f on I.
- **Extreme Value Theorem:** If f is continuous on $[a, b]$, then f has both a minimum and a maximum on $[a, b]$.
- If $f(c)$ is an extremum on an open interval containing c, then $f(c)$ is a **relative extremum.**
- Relative extrema occur only at critical numbers.
- Extrema on a closed interval occur either at critical numbers or endpoints of the interval.

First Derivative Test Let f be a continuous function on an open interval I containing c, and further let c be a critical number of f. If f is differentiable on I except possibly at c, then:

- if the sign of $f'(x)$ changes from negative to positive at c, then $f(c)$ is a relative minimum of f.
- if the sign of $f'(x)$ changes from positive to negative at c, then $f(c)$ is a relative maximum of f.

Second Derivative Test Let f be a twice differentiable function on an open interval containing c, and further let $f'(c) = 0$.

- If $f''(c) > 0$, then $f(c)$ is a relative minimum of f.
- If $f''(c) < 0$, then $f(c)$ is a relative maximum of f.
- If $f''(c) = 0$, then the test is inconclusive (use the First Derivative Test.)

Concavity

Let f be a twice differentiable function on an open interval I.

- If $f''(x) > 0$ for all x in I, the graph of f is **concave upward.**
- If $f''(x) < 0$ for all x in I, the graph of f is **concave downward.**
- If the sign of $f''(x)$ changes on either side of c, then the point $(c\,, f(c))$ is called a **point of inflection.** Furthermore, either $f''(c) = 0$ or $f''(c)$ is undefined.

Derivative of an Inverse Function

Let f be a function that is differentiable on an interval I. If f possesses an inverse function g, then g is differentiable at any x for which $f'(g(x)) \neq 0$. Moreover,

$$g'(x) = \frac{1}{f'(g(x))}, \quad f'(g(x)) \neq 0.$$

Basic Differentiation Rules

$\frac{d}{dx}[cu] = cu'$	$\frac{d}{dx}[u \pm v] = u' \pm v'$
$\frac{d}{dx}[uv] = uv' + vu'$	$\frac{d}{dx}\left[\frac{u}{v}\right] = \frac{vu' - uv'}{v^2}$
$\frac{d}{dx}[c] = 0$	$\frac{d}{dx}[u^n] = nu^{n-1}u'$
$\frac{d}{dx}[x] = 1$	$\frac{d}{dx}[\lvert u\rvert] = \frac{u}{\lvert u\rvert}(u'), \quad u \neq 0$
$\frac{d}{dx}[\ln u] = \frac{u'}{u}$	$\frac{d}{dx}[e^u] = e^u u'$
$\frac{d}{dx}[\sin u] = (\cos u)\, u'$	$\frac{d}{dx}[\cos u] = -(\sin u)\, u'$
$\frac{d}{dx}[\tan u] = \left(\sec^2 u\right) u'$	$\frac{d}{dx}[\cot u] = -\left(\csc^2 u\right) u'$
$\frac{d}{dx}[\sec u] = (\sec u \, \tan u)\, u'$	$\frac{d}{dx}[\csc u] = -(\csc u \, \cot u)\, u'$
$\frac{d}{dx}[\arcsin u] = \frac{u'}{\sqrt{1-u^2}}$	$\frac{d}{dx}[\arccos u] = \frac{-u'}{\sqrt{1-u^2}}$
$\frac{d}{dx}\left[\operatorname{arccsc} \frac{u^2-1}{\sqrt{1-u^2}}\right] = \frac{u'}{\lvert u\rvert \sqrt{u^2-1}}$	$\frac{d}{dx}[\operatorname{arccsc} u] = \frac{-u'}{\lvert u\rvert \sqrt{u^2-1}}$

Chain Rule

If $y = f(u)$ is a differentiable function of u and $u = g(x)$ is a differentiable function of x, then $y = f(g(x))$ is a differentiable function of x and

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

or equivalently,

$$\frac{d}{dx}[f(g(x))] = f'(g(x))\, g'(x).$$

4.4.3 Integration

Indefinite Integration

Antiderivatives A function F is an **antiderivative** of a function f on an interval I if $F'(x) = f(x)$ for all x in I.

The Indefinite Integral The differential equation

$$\frac{dy}{dx} = f(x)$$

may be rewritten as

$$dy = f(x)\,dx.$$

The operation of solving this equation is called **antidifferentiation** or **indefinite integration**. The general solution is denoted

$$y = \int f(x)\,dx = F(x) + C.$$

Definite Integration

Area **Summation Formulas**

$$\sum_{i=1}^{n} ka_i = k\sum_{i=1}^{n} a_i \qquad \sum_{i=1}^{n}(a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$$

$$\sum_{i=1}^{n} c = cn \qquad \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

Area of a Plane Region

If f is continuous and nonnegative on the interval $[a, b]$, the **area** of the region bounded by the graph of f, the x-axis, and the vertical lines $x = a$ and $x = b$ is

$$\text{Area} \equiv \lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\,\Delta x, \qquad x_{i-1} \le c_i \le x_i$$

where $\Delta x = (b - a)/n$.

Riemann Sums Let f be defined on $[a, b]$ and let Δ be a partition of $[a, b]$ given by $a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$, where Δx_i is the length of the ith subinterval. If c_i is any point in the ith subinterval, then the sum

$$\sum_{i=1}^{n} f(c_i)\,\Delta x_i, \qquad x_{i-1} \le c_i \le x_i$$

is called a **Riemann Sum** of f for the partition Δ. The length of the largest subinterval of a partition Δ is called the **norm** of the partition, and is denoted $\|\Delta\|$.

The Definite Integral If f is defined on $[a, b]$ and the limit

$$\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(c_i)\, \Delta x_i$$

exists, then f is **integrable** on $[a, b]$ and the limit is called the **definite integral** of f from a to b, and is denoted by

$$\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(c_i)\, \Delta x_i = \int_a^b f(x)\, dx.$$

The Definite Integral as the Area of a Region If f is continuous and nonnegative on $[a, b]$, then the area of the region bounded by the graph of f, the x-axis, and the vertical lines $x = a$ and $x = b$ is given by

$$\text{Area} = \int_a^b f(x)\, dx.$$

Area of a Region Between Two Curves If f and g are continuous on $[a, b]$ and $g(x) \leq f(x)$ for all x in $[a, b]$, then the area of the region bounded by the graphs of f and g and the vertical lines $x = a$ and $x = b$ is

$$\text{Area} = \int_a^b [f(x) - g(x)]\, dx.$$

Average Value If f is integrable on $[a, b]$, then the **average value** of f on $[a, b]$ is

$$\frac{1}{b-a} \int_a^b f(x)\, dx.$$

Volume of Solids of Revolution

- **Disc method** (R is the radius of a representative disc)

Horizontal axis of revolution	$V = \pi \int_a^b [R(x)]^2\, dx$
Vertical axis of revolution	$V = \pi \int_c^d [R(y)]^2\, dy$

- **Washer method** (R and r are the outer and inner radii, respectively, of a representative washer)

Horizontal axis of revolution	$V = \pi \int_a^b ([R(x)]^2 - [r(x)]^2)\, dx$
Vertical axis of revolution	$V = \pi \int_c^d ([R(y)]^2 - [r(y)]^2)\, dy$

- **Shell method** (p is the radius and h is the height of a representative shell)

Horizontal axis of revolution	$V = 2\pi \int_c^d p(y)h(y)\,dy$
Vertical axis of revolution	$V = 2\pi \int_a^b p(x)h(x)\,dx$

Arc Length For a smooth curve given by $y = f(x)$, the **arc length** of f between a and b is

$$s = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx.$$

For a smooth curve given by $x = g(y)$, the arc length of g between c and d is

$$s = \int_c^d \sqrt{1 + [g'(y)]^2}\,dy.$$

Basic Integration Rules

$$\int k\,f(u)\,du = k\int f(u)\,du \qquad \int [f(u) \pm g(u)]\,du = \int f(u)\,du \pm \int g(u)\,du$$

$$\int du = u + C \qquad \int u^n\,du = \frac{u^{n+1}}{n+1} + C, \quad n \neq -1$$

$$\int \frac{du}{u} = \ln|u| + C \qquad \int e^u\,du = e^u + C$$

$$\int \sin u\,du = -\cos u + C \qquad \int \cos u\,du = \sin u + C$$

$$\int \tan u\,du = -\ln|\cos u| + C \qquad \int \cot u\,du = \ln|\sin u| + C$$

$$\int \sec u\,du = \ln|\sec u + \tan u| + C \qquad \int \csc u\,du = -\ln|\csc u + \cot u| + C$$

$$\int \sec^2 u\,du = \tan u + C \qquad \int \csc^2 u\,du = -\cot u + C$$

$$\int \sec u\,\tan u\,du = \sec u + C \qquad \int \csc u\,\cot u\,du = -\csc u + C$$

$$\int \frac{du}{\sqrt{a^2 - u^2}} = \arcsin\frac{u}{a} + C \qquad \int \frac{du}{a^2 + u^2} = \frac{1}{a}\arctan\frac{u}{a} + C$$

4.4.4 Fundamental Theorem of Calculus

- **First Fundamental Theorem of Calculus**

  If f is continuous on the interval $[a, b]$ and F is an antiderivative of f on the interval $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

- **Second Fundamental Theorem of Calculus**

  If f is continuous on an open interval I containing a, then for every x in I,

$$\frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x)$$